THE LOGARITHM AS A DIRECT FUNCTION

By J. W. Bradshaw

WITH AN INTRODUCTION BY

W. F. Osgood

The student of mathematics and physics meets logarithms for the first time at an early stage. He is told that "the logarithm of a number is the exponent of the power to which a certain number, taken as the base, must be raised in order to equal the given number." The definition is purely formal. Probably the beginner has never seen a proof for the existence even of the fifth root of 2, — the square, cube, and fourth roots can be constructed geometrically, — and if he has, it is not likely that it has meant anything to him. He tacitly assumes that every positive number has a positive qth root, q being any positive integer. It is then an easy step to any rational power, and the irrational powers are thought of as limiting cases, the principle being that, whenever one wants a limit in mathematics, the limit exists.

Now all of these assumptions have been justified by rigorous ϵ-proofs in well known treatises on modern analysis.* But the general student of mathematics and physics does not read these proofs, for they are uninteresting to him; and thus the great majority of students of the Calculus never see a proof that there is such a thing as a logarithm.

It is possible, however, to supply a proof by means of elementary calculus, inclusive of the general theorems about continuous functions with which all students are familiar. How simple the analysis is appears from a casual glance at the following pages, in which Mr. Bradshaw has carried through all the details of a rigorous development of the essential properties of the Logarithm and, as it appears here, of its *Inverse*, the Exponential Function, a^x. It is hoped that this presentation may prove attractive to students who have finished a thorough course in elementary calculus. W. F. O.

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* Stolz, *Allgemeine Arithmetik*, vol. 1, ch. 8; new edition by Stolz and Gmeiner, under the title *Theoretische Arithmetik*, vol. 1, ch. 8. J. Tannery, *Fonctions d'une variable*, ch. 3. E. V. Huntington, *Grundoperationen an absoluten und komplexen Grössen*, Strassburg dissertation, 1901.

1. Definition of the Logarithm. The logarithm is commonly defined as the inverse of the exponential. It may, however, be defined as a direct function and treated quite independently. The exponential function will then appear as the inverse of the logarithm.

The logarithm of x shall here be defined as the definite integral

$$\int_1^x \frac{dx}{x}$$

for all values of the variable in the interval $0 < x < \infty$. We shall denote it by $\phi(x)$.

If we draw the curve

$$y = \frac{1}{x},$$

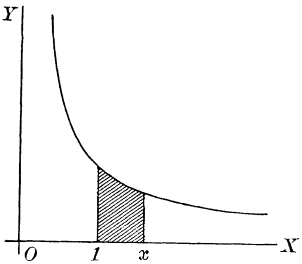

FIG. 1.

$\phi(x)$ is the area bounded by this curve, the x-axis, and two ordinates, one of which is fixed, $x = 1$, the other variable. The area is positive if the variable ordinate is to the right of the fixed ordinate, negative if to the left. If the two ordinates coincide, the area is zero, so that

$$\phi(1) = 0.$$

Since the curve $y = 1/x$ does not cut the x-axis, the area always increases as the variable ordinate moves to the right. $\phi(x)$ is therefore a single valued function of x, which everywhere increases with x.

The following theorems will serve to disclose the chief characteristics of the function.

THEOREM I. *As x increases indefinitely, $\phi(x)$ becomes positively infinite:—*

$$\phi(\infty) = \int_1^\infty \frac{dx}{x} = +\infty.$$

As x approaches zero, $\phi(x)$ becomes negatively infinite:—

$$\phi(0) = \int_1^0 \frac{dx}{x} = -\infty.$$

The proof of this theorem is ordinarily based in the Calculus on the

property of $\log x$ that $\log \infty = \infty$. Here it is necessary, of course, to give a direct proof.

Allow x to take any two values, m and l, where $m > l \geqq 1$, and note the difference in the corresponding values of $\phi(x)$.

$$\phi(m) - \phi(l) = \int_1^m \frac{dx}{x} - \int_1^l \frac{dx}{x} = \int_l^m \frac{dx}{x}.$$

By the mean value theorem (v. Picard, *Traité d'analyse*, vol. 1, p. 7),

$$\int_l^m \frac{dx}{x} = (m - l)\,\frac{1}{\mu}, \qquad l < \mu < m.$$

If we put $m = 2l$,

$$\phi(2l) - \phi(l) = l\,\frac{1}{\mu} > l/2l = 1/2.$$

Hence

$$\phi(2l) > \phi(l) + 1/2.$$

Since $\phi(1) = 0$, we have

$$\phi(2) > 1/2, \quad \phi(4) > 1, \ldots, \phi(2^n) > n/2.$$

If we have given any positive constant G, however large, and if we choose $n > 2G$, then for all values of $x > 2^n$, $\phi(x) > G$. Therefore $\phi(x)$ becomes positively infinite as x increases indefinitely, and thus the first part of the theorem is proved.

To prove the second part of the theorem, put $x = 1/x'$; then

$$\phi(x) = \int_1^x \frac{dx}{x} = -\int_1^{x'} \frac{dx}{x} = -\phi(x').$$

As x approaches zero, x' increases indefinitely and $\phi(x')$ becomes positively infinite; hence $\phi(x)$ becomes negatively infinite.

THEOREM II. *The function $\phi(x)$ is continuous in the interval $0 < x < \infty$. It has, at each point of this interval, a derivative which is also continuous and which is given by the formula:*

$$\phi'(x) = \frac{1}{x}.$$

This theorem follows at once from the theorem of the Integral Calculus that the function

$$F(x) = \int_{x_0}^{x} f(x)\, dx,$$

where $f(x)$ is continuous, is a continuous function having $f(x)$ as its derivative. The proof is given here for completeness.

We wish to prove first that, if x_0 is any positive number,

$$\lim_{x = x_0} \phi(x) = \phi(x_0)\cdot$$

Let $x_0 + \Delta x$ be any positive value for x near x_0. Then

$$\phi(x_0 + \Delta x) - \phi(x_0) = \int_1^{x_0 + \Delta x} \frac{dx}{x} - \int_1^{x_0} \frac{d}{x}$$

$$= \int_{x_0}^{x_0 + \Delta x} \frac{dx}{x} = \Delta x \, \frac{1}{x_0 + \theta \Delta x}, \qquad 0 < \theta < 1.$$

Since $x_0 + \theta \Delta x$ approaches x_0 as its limit, the limit of the second member is 0, and hence the limit of the first member is 0. Therefore

$$\lim_{\Delta x = 0} \phi(x_0 + \Delta x) = \phi(x_0),$$

and $\phi(x)$ is continuous.

Next form the difference quotient at the point x_0:

$$\frac{\phi(x_0 + \Delta x) - \phi(x_0)}{\Delta x} = \frac{1}{x_0 + \theta \Delta x}\cdot$$

Then

$$\lim_{\Delta x = 0} \frac{\phi(x_0 + \Delta x) - \phi(x_0)}{\Delta x} = \phi'(x_0) = \frac{1}{x_0}\cdot$$

Hence $\phi(x)$ has a derivative, $1/x$, and this is continuous throughout the interval $0 < x < \infty$.

To sum up, we have shown that $\phi(x)$ is a function of x which is single-valued and continuous in the interval $0 < x < \infty$, which always increases with x, and which has at each point of the interval a positive continuous derivative,

$\phi'(x) = 1/x$; that $\phi(1) = 0$, $\phi(0) = -\infty$, and $\phi(\infty) = \infty$. The graph of the function is therefore as shown in the figure.

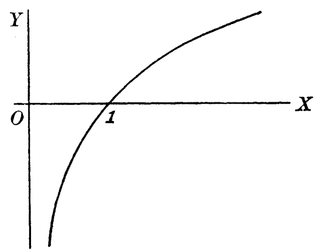

FIG. 2.

2. The Inverse Function. THEOREM. *The equation*

$$y = \phi(x),$$

x lying in the interval $0 < x < \infty$, defines x as a function of y:

$$x = \psi(y)$$

which is single valued and continuous in the interval $-\infty < y < \infty$, increases with y, and has, at each point of the interval, a continuous positive derivative.

The function $\psi(y)$ will turn out to be e^y. We wish to prove first that, any value y_0 being given, there exists one and only one value x_0 of x satisfying the equation*

$$y_0 = \phi(x). \tag{1}$$

Geometrically this means that the line $y = y_0$ cuts the curve $y = \phi(x)$ in one and only one point. An examination of the figure shows at once that this is the case, and as an intuitive proof this is sufficient. An analytic proof may be given by means of the following well known theorem of continuous functions.

As x varies continuously from a to b, any function $f(x)$, continuous in the interval $a \leqq x \leqq b$, passes through all values included between $f(a)$ and $f(b)$.

Since $\phi(0) = -\infty$ and $\phi(\infty) = +\infty$, $\phi(x)$ must have taken on the value y_0 for some positive value x_0 of x. Equation (1) has therefore one solution, x_0. It has, moreover, only one solution. Suppose there were a second positive solution x_1. This value cannot be greater than x_0, for, if it were, $\phi(x_1)$ would be greater than $\phi(x_0)$, since $\phi(x)$ always increases with x. Similarly x_1 cannot be less than x_0. Hence x is a single valued function of y in the interval $-\infty < y < \infty$. Call it $\psi(y)$.

If y_0, y_1 are any two values of y, and $y_0 < y_1$, then evidently $\psi(y_0) < \psi(y_1)$.

To show that $x = \psi(y)$ is a continuous function of y, take an arbitrary

* This theorem is proven in advanced treatises on the Differential Calculus; *cf.* Stolz. *Differential- und Integralrechnung*, vol. 1, p. 38.

value y_0 of y and an auxiliary interval

$$P: \quad 0 < a < x < b$$

so as to include the point $x_0 = \psi(y_0)$.

Let $\phi(a) = A$ and $\phi(b) = \text{B}$. Then $A < y_0 < B$. Let $y_0 + \Delta y$ be any second value of y lying between A and B, and let $x_0 + \Delta x$ be the corresponding value of x; it must be in the interval P. We wish to show that when Δy approaches 0, Δx also approaches 0.

$$y_0 = \phi(x_0), \qquad y_0 + \Delta y = \phi(x_0 + \Delta x),$$

$$\begin{aligned} \Delta y &= \phi(x_0 + \Delta x) - \phi(x_0), \\ &= \Delta x\, \phi'(x_0 + \theta\Delta x), \qquad 0 < \theta < 1, \\ &= \Delta x \frac{1}{x_0 + \theta\Delta x}. \end{aligned}$$

But $x_0 + \theta\Delta x$ lies in the interval P, hence

$$|\Delta y| > |\Delta x| \frac{1}{b}, \qquad |\Delta x| < b|\Delta y|.$$

Therefore Δx approaches 0 when Δy does, and $\psi(y)$ is continuous at the point y_0, which was any point.

Furthermore $\psi(y)$ has a continuous derivative. For the difference quotient is given by the formula

$$\frac{\Delta x}{\Delta y} = x_0 + \theta\Delta x \,.$$

Since Δx approaches 0 when Δy does,

$$\lim_{\Delta y = 0} \frac{\Delta x}{\Delta y} = x_0 \,.$$

Hence $\psi(y)$ has a derivative

$$\psi'(y) = \psi(y),$$

and this derivative is continuous.

3. The Fundamental Property of Logarithms. We will now prove the fundamental property of logarithms, that the sum of the logarithms of two positive numbers is equal to the logarithm of their product. We have to prove that at each point of the region

$$R: \begin{cases} 0 < x < \infty \\ 0 < y < \infty \end{cases}$$

the functional equation

$$\phi(x) + \phi(y) = \phi(xy)$$

is satisfied. To do this, write the left hand side of the equation in the form

$$\int_1^x \frac{dt}{t} + \int_1^y \frac{dt}{t}$$

and then replace the variable of integration in the second integral by t', where*

$$t' = xt.$$

We have, then,

$$\int_1^y \frac{dt}{t} = \int_x^{xy} \frac{dt'}{t'}.$$

Hence, dropping the accent in this last integral and substituting above, we obtain

$$\int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} = \int_1^{xy} \frac{dt}{t},$$

or $\phi(xy)$, as the value of $\phi(x) + \phi(y)$, and this is the result which we set out to establish.

4. A Second Property of the Function $\phi(x)$. Existence of Roots. From the law

$$\phi(x) + \phi(y) = \phi(xy) \tag{2}$$

we will now deduce another property of the function ϕ, namely,

$$\phi(x^\alpha) = \alpha\phi(x), \tag{3}$$

where α is any rational number. Incidentally a proof is obtained of the existence of a positive qth root of any positive number a and hence of a positive value for every commensurable power of a : $a^{p/q}$.

Equation (2) gives us at once the following facts with regard to the function ϕ:

$$\phi(1/x) + \phi(x) = \phi(1) = 0,$$

$$\therefore\ \phi(1/x) = -\phi(x). \tag{4}$$

* This method has been employed in the study of the logarithmic function for complex values of the argument by Burkhardt, *Analytische Funktionen*, p. 162.

$$\phi(x) + \phi(x) + \cdots \text{ to } n \text{ terms} = \phi(x \cdot x \cdot \cdots \text{ to } n \text{ factors}),$$

$$\therefore\ n\phi(x) = \phi(x^n), \tag{5}$$

n being a positive integer.

Assuming, as we have in the last equation, the definition of a^n when n is a positive integer, we can prove the

THEOREM. *Every positive number a has one, and only one, positive qth root, q being a positive integer.*

Suppose a has a positive qth root, x; then

$$a = x^q,$$

$$\phi(a) = \phi(x^q) = q\phi(x).$$

Let $\phi(a) = b$, and we have

$$x = \psi\left(\frac{b}{q}\right). \tag{6}$$

This is a necessary relation for any positive qth root of a, and it shows, too, that if a has a positive qth root it can have but one, since ψ is a single valued function. Further, we know there is a positive number x satisfying (6), and, retracing our steps, the qth power of x is a. Therefore the number x determined by (6) is a positive qth root of a, and the only one.

If we define $a^{p/q}$ as $(\sqrt[q]{a})^p$* and $a^{-p/q}$ as $\dfrac{1}{(\sqrt[q]{a})^p}$, the theorem shows that each of these expressions has one and only one positive value. If further a^0 is defined as 1, a^α, assumed positive, is defined and single valued for all rational values of α.

If in (5) we put $y = x^n$, we have

$$n\phi(y^{\frac{1}{n}}) = \phi(y),$$

$$\therefore\quad \phi(y^{\frac{1}{n}}) = \frac{1}{n}\phi(y). \tag{7}$$

$$\phi(y^{\frac{1}{n}}) + \phi(y^{\frac{1}{n}}) + \cdots \text{ to } m \text{ terms} = \frac{m}{n}\phi(y),$$

and also

$$= \phi[(y^{\frac{1}{n}})^m],$$

$$\therefore\quad \phi(y^{\frac{m}{n}}) = \frac{m}{n}\phi(y). \tag{8}$$

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* It will appear later that $(\sqrt[q]{a})^p = \sqrt[q]{a^p}$.

From (9)

$$\phi\left(\frac{1}{y^{\frac{m}{n}}}\right) = -\,\phi(y^{\frac{m}{n}}),$$

$$\therefore\ \phi(y^{-\frac{m}{n}}) = -\frac{m}{n}\,\phi(y)\cdot \tag{9}$$

Finally, since

$$\phi(x^0) = \phi(1) = 0, \tag{10}$$

we have, for all commensurable numbers α, the relation

$$\phi(x^\alpha) = \alpha\phi(x). \tag{3}$$

From this law may be deduced at once as corollaries the following theorems :

1°. $$\sqrt[q]{a^p} = (\sqrt[q]{a})^p.$$

2°. $$\sqrt[mq]{a^{mp}} = \sqrt[q]{a^p}.$$

3°. $$\sqrt[q]{\sqrt[m]{a}} = \sqrt[mq]{a}.$$

4°. $$\sqrt[q]{a^p}.\sqrt[q]{b^p} = \sqrt[q]{(ab)^p}.$$

5. Definition of the Exponential Function. Give to x any fixed positive value a and replace α by x; let $\phi(a) = b$; (3) then becomes

$$\phi(a^x) = xb, \tag{11}$$

or

$$a^x = \psi(xb), \tag{12}$$

a relation which holds for all rational values of x. We shall define a^x for irrational values of x by this relation. It follows that a^x is a single valued, positive, continuous function of x, and that it has a continuous derivative

$$\frac{da^x}{dx} = \frac{d}{dx}\,\psi(xb) = b\psi'(xb) = b\psi(xb) = a^x\phi(a)\cdot$$

The laws of indices may be derived very simply from (11). Let x and y be any real numbers and a and b positive numbers.

1) $$a^x a^y = a^{x+y}. \tag{13}$$

For

$$\phi(a^x a^y) = \phi(a^x) + \phi(a^y)$$
$$= x\phi(a) + y\phi(a) = (x+y)\phi(a) = \phi(a^{x+y}).$$
$$\therefore\ a^x a^y = a^{x+y}.$$

Similarly it is shown that

$$
\begin{aligned}
&2) \qquad (a^x)^y = a^{xy}, \\
&3) \qquad a^x b^x = (ab)^x.
\end{aligned}
\tag{14}
$$

6. Two Functional Relations for the Function $\psi(x)$. If we define the number e by the equation

$$\phi(e) = 1,$$

equation (12) becomes

$$e^x = \psi(x).$$

Using the special value e for a in equation (13), we have the first functional relation

$$\psi(x) \cdot \psi(y) = \psi(x + y). \tag{15}$$

Substituting e for a in (14) we have a second functional relation

$$\psi(xy) = [\psi(x)]^y. \tag{16}$$

7. The Sufficiency of the Functional Relations. It turns out that the functional relations (2), (15) are sufficient for the determination of the functions, as is stated more precisely in the following theorems.

THEOREM. *$\phi(x)$ is the only function of x, single valued and continuous for all positive values of x, which satisfies the functional relation*

$$\Phi(x) + \Phi(y) = \Phi(xy) \tag{2}$$

and takes on the value 1 when $x = e$:

Suppose there were a second function, $\Phi(x)$, meeting all these requirements. Equation (2) shows that $\Phi(1)= 0$. Then by the reasoning by which (11) was derived from (2),

$$\Phi(a^x) = x\Phi(a),$$

when x is rational and hence, because of the continuity of the function Φ, for all values of x; or e being substituted for a,

$$\Phi(e^x) = x\ \Phi(e) = x = \phi(e^x).$$

Since e^x takes on all positive values, $\Phi(x)$ and $\phi(x)$ are equal for all positive values of x.

The function $\phi(x)$ thus turns out to be the natural logarithm of x.

A similar theorem holds regarding the function $\psi(x)$.

THEOREM. *$\psi(x)$ is the only single valued, positive, continuous function of x which satisfies the relation*

$$\Psi(x) \cdot \Psi(y) = \Psi(x + y) \tag{15}$$

and which has the value e when $x = 1$.

By a course of reasoning similar to that by which (11) was derived from (2), it may be shown that any function Ψ which satisfies the conditions of the theorem satisfies also the relation

$$\Psi(xy) = [\Psi(y)]^x,$$

x being any rational number, and hence, because of the continuity, any number whatever. Put $y = 1$ and we have

$$\Psi(x) = e^x = \psi(x).$$

8. The Logarithm to any Base. The function $\phi(x)$ which we have been considering is the logarithm to the base e. The logarithm to any other base, a, may be defined thus:

$$\log_a x = \frac{\phi(x)}{\phi(a)}.$$

Any positive number except 1 can be used as base. It will be seen that this function possesses the chief properties of $\phi(x)$; $\log_a x$ is a single valued, continuous function for all positive values of x, has a continuous derivative $\frac{1}{x\phi(a)}$, and satisfies the fundamental law

$$\log_a x + \log_a y = \log_a xy.$$

The following relations are often useful.

1°. If $$a^x = b, \qquad a \neq 1,$$

then $$\phi(a^x) = x\phi(a) = \phi(b),$$

$$x = \frac{\phi(b)}{\phi(a)},$$

and therefore $$b = a^{\log_a b}.$$

2°. If $$b = a^x = c^{xy}, \qquad a, c \neq 1,$$

then $$y = \frac{\phi(a)}{\phi(c)}, \qquad xy = \frac{\phi(b)}{\phi(c)},$$

whence

$$x = \frac{\phi(b)}{\phi(c)} \div \frac{\phi(a)}{\phi(c)} = \frac{\log_c b}{\log_c a},$$

and therefore

$$\log_a b = \frac{\log_c b}{\log_c a}.$$

HARVARD UNIVERSITY,
JUNE, 1902.

ON POSITIVE QUADRATIC FORMS

BY PAUL SAUREL

THE necessary and sufficient conditions that a homogeneous quadratic function of n variables be constantly positive or constantly negative are well known. A very simple demonstration of the necessity of these conditions has been given by Gibbs in his great memoir On the Equilibrium of Heterogeneous Substances.* This demonstration, however, has not received the attention which it deserves, perhaps because its simplicity is somewhat disguised by the physical terms employed. In the present note we shall reproduce Gibbs's demonstration and we shall complete it by showing that certain of the conditions thus obtained are sufficient.

Let us consider the quadratic function ϕ defined by the equation

$$\phi = \sum_{i=1}^{n} \sum_{k=1}^{n} a_{ik} x_i x_k, \tag{1}$$

in which

$$a_{ik} = a_{ki}, \tag{2}$$

and let us write

$$f_i = \sum_{k=1}^{n} a_{ik} x_k. \tag{3}$$

From (3) we get

$$df_i = \sum_{k=1}^{n} a_{ik} dx_k, \tag{4}$$

* *Transactions of the Connecticut Academy of Arts and Sciences*, vol. 3, part 1, page 166 (1876).